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On the maximum area pentagon in a planar point set[☆]

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Abstract

A finite set of points in the plane is described as in convex position if it forms the set of vertices of a convex polygon. This work studies the ratio between the maximum area of convex pentagons with vertices in P and the area of the convex hull of P, where the planar point set P is in convex position.

Keywords: Convex position; Convex hull; Affine transformation

1. Introduction

Ref. [2] shows that in the study of motion-planning problems in robotics by using heuristics, the largest area polygons in a planar point set play an important role. Refs. [7,8] and [9] discuss these problems and contain the related results.

A finite set of points in the plane is described as in *convex position* if it forms the set of vertices of a convex polygon. Let P be a finite set of points in convex position in the plane; hence any subset of P is also a point set in convex position. Denote the area of the convex hull of $Q \subset P$ by $S(Q)$. For the sake of convenience we may call a subset $Q \subset P$ a polygon if Q forms the vertices of a polygon. Let

$$f_k(P) := \max\left\{\frac{S(Q)}{S(P)} : Q \subset P,\ P \text{ is in convex position}\right\}$$
$$f_k^{\text{conv}}(n) := \min\{f_k(P) : |P| = n,\ P \text{ is in convex position}\}.$$

Ref. [1] mainly studies $f_3^{\text{conv}}(n)$. In this work we evaluate $f_5^{\text{conv}}(n)$.

Instead of considering the ratio between the area of the convex hull of a point set and the area of the convex hull of its subset, [4–6] study the quantitative Steinitz Theorem and prove that any set whose convex hull contains a disk

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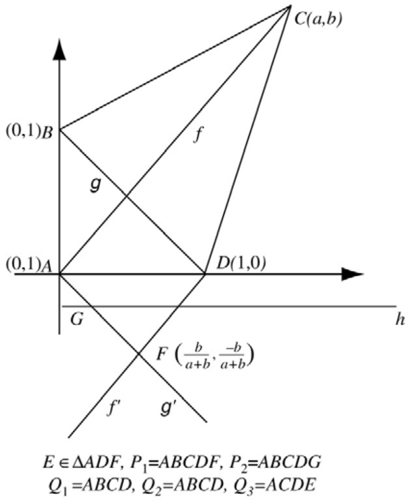

Fig. 1.

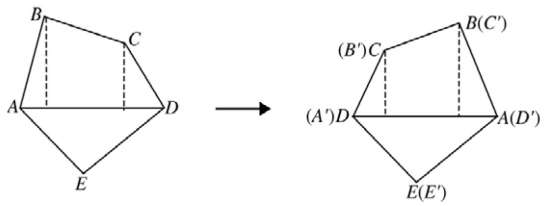

Fig. 2.

with center O and radius 1 has a subset of at most four points whose convex hull contains a disk with center O and radius $r_4 = \frac{\cos\frac{2\pi}{3}}{\cos\frac{\pi}{5}}$.

2. Main results

Lemma 1. $f_4^{\text{conv}}(5) = \frac{2}{5-\sqrt{5}}$.

Proof. Let P be a convex 5-gon with vertices A, B, C, D, E in clockwise order. Suppose the 4-gon $ABCD$ is a maximum area 4-gon in P. Given two triangles, there exists a unique affine transformation which transforms one triangle into another. So, without loss of generality we assume that $A = (0, 0)$, $B = (0, 1)$, $D = (1, 0)$, $C = (a, b)$ $(a > 0, b > 0)$. Let $b \geq 1$; see Fig. 1. Indeed, when $b < 1$, the distance from B to the straight line AD is greater than the distance from C to the straight line AD, and we can reflect P about a vertical line, which does not change the ratio of the areas.

See Fig. 2. Relabel the vertices of P to ensure that the distance from C' to the straight line $A'D'$ is greater than the distance from B' to the straight line $A'D'$, and in this way we come to the case of $b \geq 1$.

Let Q_1, Q_2, Q_3 denote 4-gons $ABCD$, $ABDE$, $ACDE$ respectively. Let f be the line through A and C, and f' be the parallel line through D. Similarly, let g be the line through B and D, and g' be the parallel line through A. For Q_1 to be the maximum area 4-gon in P, E must lie completely above f' and g'. Define $F = f' \cap g'$; then $F = (\frac{b}{a+b}, \frac{-b}{a+b})$ and $E \in \triangle ADF$, and hence P is always contained in the convex 5-gon $P_1 = ABCDF$. Since $b \geq 1$, $S(Q_3) \geq S(Q_2)$; and since $S(Q_1) \geq S(Q_3)$, $S(\triangle ABC) \geq S(\triangle ADE)$. Suppose $E = (x_0, y_0)$,

$$S(\triangle ABC) = \frac{a}{2}, S(\triangle ADE) = \frac{-y_0}{2} \Rightarrow \frac{a}{2} \geq \frac{-y_0}{2} \Rightarrow y_0 \geq -a.$$

Then E lies above the horizontal line $h : y = -a$.

Case 1. Suppose F lies above the line h; then $\frac{-b}{a+b} \geq -a$ (that is $\frac{b}{a+b} \leq a$). Notice that $P \subset P_1$ and so $S(P) \leq S(P_1)$;

$$S(Q_1) = \frac{1}{2}(a+b), \quad S(P_1) = \frac{1}{2}(a+b) + \frac{1}{2}\left(\frac{b}{a+b}\right). \tag{$*$}$$

Subcase 1.1. Suppose $\frac{b}{a+b} \leq \frac{a}{b}$; then

$$\frac{1}{\frac{a}{b}+1} \leq \frac{a}{b} \Rightarrow \frac{a}{b} \geq \frac{\sqrt{5}-1}{2},$$
$$\frac{S(P)}{S(Q_1)} \leq \frac{S(P_1)}{S(Q_1)} = 1 + \frac{b}{(a+b)^2} \leq 1 + \frac{b^2}{(a+b)^2} = 1 + \frac{1}{(\frac{a}{b}+1)^2} \leq \frac{5-\sqrt{5}}{2},$$
$$\frac{S(Q_1)}{S(P)} \geq \frac{2}{5-\sqrt{5}}.$$

Subcase 1.2. Suppose $\frac{a}{b} < \frac{b}{a+b} \leq a$; then

$$\frac{1}{\frac{a}{b}+1} > \frac{a}{b} \Rightarrow \frac{a}{b} < \frac{\sqrt{5}-1}{2} \Rightarrow \frac{b}{a} > \frac{2}{\sqrt{5}-1},$$
$$\frac{S(P)}{S(Q_1)} \leq \frac{S(P_1)}{S(Q_1)} \leq 1 + \frac{a}{a+b} = 1 + \frac{1}{(\frac{b}{a}+1)} \leq \frac{5-\sqrt{5}}{2},$$
$$\frac{S(Q_1)}{S(P)} \geq \frac{2}{5-\sqrt{5}}.$$

Case 2. Suppose F lies below the line h; then $\frac{-b}{a+b} < -a$ (that is $\frac{b}{a+b} > a$), so $S(P) \leq S(P_2)$, where $P_2 = ABCDG$ is a 5-gon with $G = g' \cap h$. Since $g' : y = -x$, $h : y = -a$, $G = (a, -a)$;

$$S(P_2) = \frac{1}{2}(a+b) + \frac{1}{2}a.$$
$$\frac{b}{a+b} > a \geq \frac{a}{b} \Rightarrow \frac{b}{a+b} > \frac{a}{b} \Rightarrow \frac{b}{a} > \frac{2}{\sqrt{5}-1},$$
$$\frac{S(P)}{S(Q_1)} \leq \frac{S(P_2)}{S(Q_1)} = 1 + \frac{a}{a+b} = 1 + \frac{1}{(\frac{b}{a}+1)} \leq \frac{5-\sqrt{5}}{2},$$
$$\frac{S(Q_1)}{S(P)} \geq \frac{2}{5-\sqrt{5}}.$$

From the above argument, we obtain that for any five-point set P in convex position we have $f_4(P) \geq \frac{2}{5-\sqrt{5}}$ and hence $f_4^{\text{conv}}(5) \geq \frac{2}{5-\sqrt{5}}$.

Let $a = \frac{\sqrt{5}-1}{2}$, $b = 1$; hence the line h passes through F. Let $E = F$; then $\frac{S(Q_1)}{S(P)} = \frac{a+b}{a+b+\frac{b}{a+b}} = \frac{2}{5-\sqrt{5}}$ by ($*$), so $f_4^{\text{conv}}(5) \leq \frac{2}{5-\sqrt{5}}$.

Hence $f_4^{\text{conv}}(5) = \frac{2}{5-\sqrt{5}}$. □

Theorem 2. $f_5^{\text{conv}}(6) \geq \frac{10}{15-\sqrt{5}}$.

Proof. Let P be a convex 6-gon with vertices A, B, C, D, E, F in clockwise order. Suppose the 5-gon $Q = ABCDE$ is the maximum area 5-gon in P. Assume (by an affine transformation) that $A = (0, 0)$, $B = (0, 1)$, $E = (1, 0)$, $C = (a, b)$ and $D = (c, d)$ $(a, b, c > 0, d \geq 1)$. See Fig. 3.

Let f be the line through A and D, and f' be the parallel line through E. Similarly, let g be the line through B and E, and g' be the parallel line through A. For Q to be the maximum area 5-gon in P, F must lie completely above f'

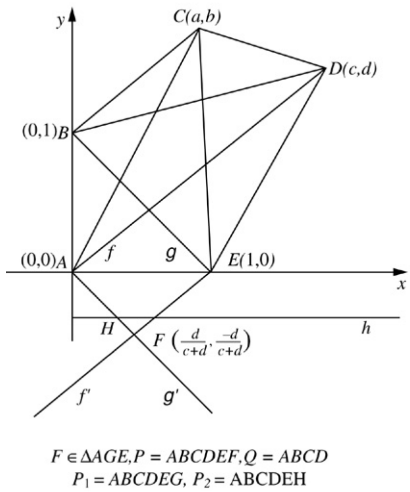

Fig. 3.

and g'. Define $G = f' \cap g'$; then $G = (\frac{d}{c+d}, \frac{-d}{c+d})$ and $F \in \triangle AGE$, and hence P is always contained in the convex 6-gon $P_1 = ABCDEG$.

Let $\mathbb{T} = \{\triangle ABC, \triangle BCD, \triangle CDE, \triangle DEA, \triangle EAB\}$, that is, $\mathbb{T}$ is the set of all triangles formed by three consecutive vertices of Q. Since $d \geq 1$, $S(\triangle DEA) \geq S(\triangle EAB)$. Without loss of generality, let $\triangle BCD$ be the triangle of $\mathbb{T}$ with the minimum area and let $S(\triangle BCD) = \frac{\alpha}{2}$; then $ABDE$ is the maximum area 4-gon in Q;

$$S(Q) = \frac{1}{2}(c + d + \alpha), \quad S(ABDE) = \frac{1}{2}(c + d).$$

By Lemma 1,

$$\frac{S(ABDE)}{S(Q)} = \frac{c+d}{c+d+\alpha} = 1 - \frac{\alpha}{c+d+\alpha} \geq \frac{2}{5-\sqrt{5}}$$
$$\frac{\alpha}{c+d+\alpha} \leq \frac{3-\sqrt{5}}{5-\sqrt{5}}. \tag{1}$$

Let $F = (x_0, y_0)$; since Q is the maximum area 5-gon in P,

$$S(\triangle BCD) \geq S(\triangle AEF) \Longrightarrow \frac{\alpha}{2} \geq \frac{-y_0}{2} \Longrightarrow y_0 \geq -\alpha.$$

So F lies above the horizontal line $h : y = -\alpha$.

Case 1. Suppose G lies above the line h; then $\frac{d}{c+d} \leq \alpha$. Notice that $P \subset P_1$ and so $S(P) \leq S(P_1)$;

$$S(P_1) = \frac{1}{2}(c + d + \alpha) + \frac{1}{2}\left(\frac{d}{c+d}\right).$$

Hence by (1)

$$\frac{S(P)}{S(Q)} \leq \frac{S(P_1)}{S(Q)} = 1 + \frac{\frac{d}{c+d}}{c+d+\alpha} \leq 1 + \frac{\alpha}{c+d+\alpha} = 1 + \frac{3-\sqrt{5}}{5-\sqrt{5}} = \frac{15-\sqrt{5}}{10},$$
$$\frac{S(Q)}{S(P)} \geq \frac{10}{15-\sqrt{5}}.$$

Case 2. Suppose G lies below the line h; then $\frac{d}{c+d} > \alpha$. So $S(P) \leq S(P_2)$, where $P_2 = ABCDEH$ is a 6-gon with $H = g' \cap h$. Since $g' : y = -x$, $h : y = -\alpha$, $H = (\alpha, -\alpha)$;

$$S(P_2) = S(Q) + S(\triangle AHE) = \frac{1}{2}(c + d + \alpha) + \frac{\alpha}{2},$$

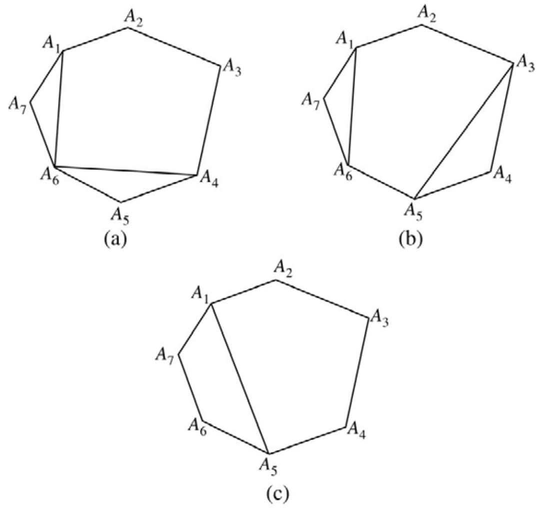

Fig. 4.

$$\frac{S(P)}{S(Q)} \leq \frac{S(P_2)}{S(Q)} = 1 + \frac{\alpha}{c+d+\alpha}.$$

By the same argument as in case 1,

$$\frac{S(Q)}{S(P)} \geq \frac{10}{15-\sqrt{5}}.$$

From the above argument, we obtain that for any six-point set P in convex position we have $f_5(P) \geq \frac{10}{15-\sqrt{5}}$ and hence $f_5^{\text{conv}}(6) \geq \frac{10}{15-\sqrt{5}}$. □

Theorem 3. $f_5^{\text{conv}}(7) \geq \frac{5}{10-\sqrt{5}}$.

Proof. Let P be a convex 7-gon with vertices A_1, A_2, A_3, A_4, A_5, A_6, A_7 in clockwise order. Suppose 5-gon Q is the maximum area 5-gon in P; then Q must be in one of the forms $A_iA_{i+1}A_{i+2}A_{i+3}A_{i+5}$, $A_iA_{i+1}A_{i+2}A_{i+4}A_{i+5}$, $A_iA_{i+1}A_{i+2}A_{i+3}A_{i+4}$ (the addition in the subscript is modulo 7).

Case 1. Suppose $Q = A_iA_{i+1}A_{i+2}A_{i+3}A_{i+5}$. Without loss of generality let $Q = A_1A_2A_3A_4A_6$, as shown in Fig. 4(a). Let $P_1 = A_1A_2A_3A_4A_5A_6$, $P_2 = A_1A_2A_3A_4A_6A_7$. Then Q is also the maximum area 5-gon in P_1 and in P_2. By Theorem 2, we have

$$\frac{S(P)}{S(Q)} = \frac{S(P_1)+S(P_2)-S(Q)}{S(Q)} \leq \frac{15-\sqrt{5}}{10} + \frac{15-\sqrt{5}}{10} - 1 = \frac{10-\sqrt{5}}{5}.$$

Thus

$$\frac{S(Q)}{S(P)} \geq \frac{5}{10-\sqrt{5}}.$$

Case 2. Suppose $Q = A_iA_{i+1}A_{i+2}A_{i+4}A_{i+5}$; see Fig. 4(b). By the same argument as in case 1 the same conclusion is obtained.

Case 3. Suppose $Q = A_iA_{i+1}A_{i+2}A_{i+3}A_{i+4}$, that is, Q is formed by five consecutive vertices of P. Without loss of generality, let $Q = A_1A_2A_3A_4A_5$, as shown in Fig. 4(c). Assume that (via an affine transformation) $A_1 = (0,0)$, $A_2 = (0,1)$, $A_5 = (1,0)$, $A_3 = (a,b)$, $A_4 = (c,d)$ $(a,b,c > 0, d \geq 1)$. See Fig. 5. Let f be the line through A_1 and A_4, and f' be the parallel line through A_5. Similarly, let g be the line through A_2 and A_5, and let g' be the parallel line through A_1. Let $G = f' \cap g'$, so $G = (\frac{d}{c+d}, \frac{-d}{c+d})$. Like in the proof of Theorem 2, here $A_6, A_7 \in \triangle A_1A_5G$ and P is always contained in the convex 6-gon $P_1 = A_1A_2A_3A_4A_5G$.

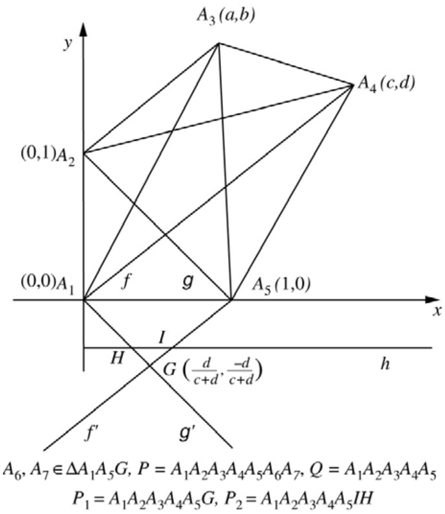

Fig. 5.

Let $\mathbb{T} = \{\triangle A_i A_{i+1} A_{i+2} \mid i = 1, 2, \ldots, 5\}$ (the addition in the subscript is modulo 5). Suppose $\triangle A_2A_3A_4$ is the triangle of $\mathbb{T}$ with the minimum area and let $S(\triangle A_2A_3A_4) = \frac{\alpha}{2}$; then $A_1A_2A_4A_5$ is the maximum area 4-gon in Q. Like in the proof of Theorem 2, A_6, A_7 lies above the horizontal line $h : y = -\alpha$;

$$S(Q) = \frac{1}{2}(c + d + \alpha), \quad S(A_1A_2A_4A_5) = \frac{1}{2}(c + d).$$

By Lemma 1,

$$\frac{S(A_1A_2A_4A_5)}{S(Q)} = \frac{c+d}{c+d+\alpha} = 1 - \frac{\alpha}{c+d+\alpha} \geq \frac{2}{5-\sqrt{5}}$$

$$\frac{\alpha}{c+d+\alpha} \leq \frac{3-\sqrt{5}}{5-\sqrt{5}}. \tag{2}$$

Subcase 3.1. Suppose G lies above the line h; then $\frac{d}{c+d} \leq \alpha$. By the same argument as in case 1 of Theorem 2,

$$\frac{S(P)}{S(Q)} \leq \frac{S(P_1)}{S(Q)} = 1 + \frac{\frac{d}{c+d}}{c+d+\alpha} \leq 1 + \frac{\alpha}{c+d+\alpha} = 1 + \frac{3-\sqrt{5}}{5-\sqrt{5}} = \frac{15-\sqrt{5}}{10},$$

$$\frac{S(Q)}{S(P)} \geq \frac{10}{15-\sqrt{5}}.$$

Subcase 3.2. Suppose G lies below the line h; $\frac{d}{c+d} > \alpha$. Then P must be contained in the 7-gon $P_2 = A_1A_2A_3A_4A_5IH$, where $I = f' \cap h$, $H = g' \cap h$. Since $f' : y = \frac{d}{c}(x-1)$, $g' : y = -x$, $h : y = -\alpha$, $I = (1 - \frac{\alpha c}{d}, -\alpha)$, $H = (\alpha, -\alpha)$.

$$S(P_2) = S(Q) + S(A_1A_2IH) = \frac{1}{2}(c+d+\alpha) + \frac{1}{2}\left(2\alpha - \frac{\alpha^2(c+d)}{d}\right).$$

Hence by (2)

$$\begin{aligned} \frac{S(P)}{S(Q)} &\leq \frac{S(P_2)}{S(Q)} = 1 + \frac{2\alpha}{c+d+\alpha} - \frac{\frac{\alpha^2(c+d)}{d}}{c+d+\alpha} \\ &< 1 + \frac{2\alpha}{c+d+\alpha} \leq 1 + 2\left(\frac{3-\sqrt{5}}{5-\sqrt{5}}\right) = \frac{15-\sqrt{5}}{10}, \end{aligned}$$

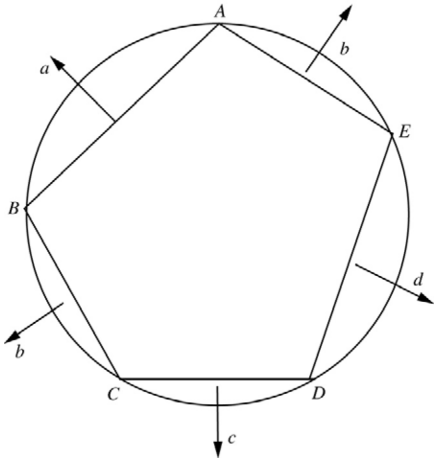

Fig. 6.

$$\frac{S(Q)}{S(P)} \geq \frac{10}{15-\sqrt{5}}.$$

From the above argument, we obtain that for any seven-point set P in convex position we have $f_5(P) \geq \frac{5}{10-\sqrt{5}}$ and hence $f_5^{\text{conv}}(7) \geq \frac{5}{10-\sqrt{5}}$. $\square$

Lemma 4. *Let P_n be the set of vertices of a regular n-gon, and let $r_5(n) := f_5(P_n)$; then*

$$r_5(n) = \frac{5\sin\frac{2\pi}{5}}{n\sin\frac{2\pi}{n}} \quad \textit{when } n \equiv 0 \pmod 5;$$

$$r_5(n) = \frac{4\sin\frac{\lfloor\frac{n}{5}\rfloor 2\pi}{n} + \sin\frac{\lceil\frac{n}{5}\rceil 2\pi}{n}}{n\sin\frac{2\pi}{n}} \quad \textit{when } n \equiv 1 \pmod 5;$$

$$r_5(n) = \frac{3\sin\frac{\lfloor\frac{n}{5}\rfloor 2\pi}{n} + 2\sin\frac{\lceil\frac{n}{5}\rceil 2\pi}{n}}{n\sin\frac{2\pi}{n}} \quad \textit{when } n \equiv 2 \pmod 5;$$

$$r_5(n) = \frac{2\sin\frac{\lfloor\frac{n}{5}\rfloor 2\pi}{n} + 3\sin\frac{\lceil\frac{n}{5}\rceil 2\pi}{n}}{n\sin\frac{2\pi}{n}} \quad \textit{when } n \equiv 3 \pmod 5;$$

$$r_5(n) = \frac{\sin\frac{\lfloor\frac{n}{5}\rfloor 2\pi}{n} + 4\sin\frac{\lceil\frac{n}{5}\rceil 2\pi}{n}}{n\sin\frac{2\pi}{n}} \quad \textit{when } n \equiv 4 \pmod 5.$$

Proof. Suppose that the maximum area 5-gon *ABCDE* with vertices in P_n divides the boundary of the convex hull of P_n into five chains *AB*, *BC*, *CD*, *DE* and *EA*, with a, b, c, d and e edges, respectively, as shown in Fig. 6. First, we prove that *any two of these numbers differ by at most 1*.

Case 1. Suppose for *two adjacent numbers*, say, a and b, we have $a - b \geq 2$. See Fig. 7. Let F be the nearest point of P_n to B in clockwise order. Observe that since $a - b \geq 2$, the number of points of P_n on $\widehat{AB}$ is at least two more than that of points on $\widehat{BC}$. Then $S(\triangle AFC) > S(\triangle ABC)$, and the area of the 5-gon *AFCDE* is greater than the area of the 5-gon *ABCDE*, contradicting the choice of the 5-gon *ABCDE*.

Case 2. Suppose that for *two nonadjacent numbers*, say, a and c, we have $a - c \geq 2$. From case 1, we only need to consider the case $a = k, b = k - 1, c = k - 2$. See Fig. 8. Let G be the nearest point of P_n to B and F be the nearest point of P_n to C in clockwise order; then $S(ABCDE) = S(ABFDE) < S(AGFDE)$, contradicting the choice of the 5-gon *ABCDE*.

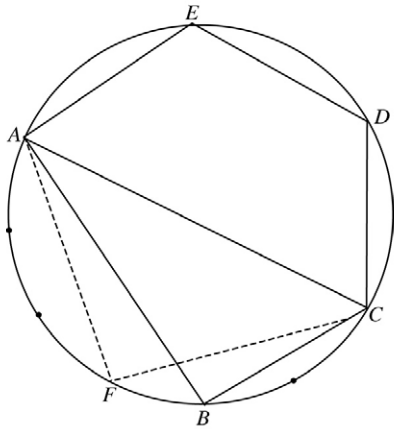

Fig. 7.

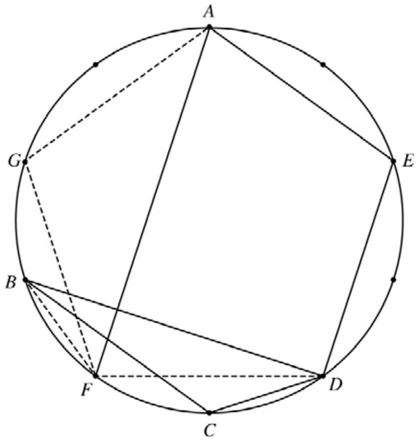

Fig. 8.

Therefore, we conclude that the maximal area 5-gon splits the boundary into five chains whose numbers of edges are $\{t, t, t, t, t\}$, $\{t, t, t, t, t + 1\}$, $\{t, t, t, t + 1, t + 1\}$, $\{t, t, t + 1, t + 1, t + 1\}$, $\{t, t + 1, t + 1, t + 1, t + 1\}$, when $n \equiv 0, 1, 2, 3, 4 \bmod 5$, respectively. An easy computation leads to the claimed formulas. $\square$

Notice that each $r_5(n)$ is a decreasing function. Thus we can deduce that

$$\lim_{n\to\infty} r_5(n) = \frac{5}{2\pi}\sin\frac{2\pi}{5} = \frac{5\sqrt{10+2\sqrt{5}}}{8\pi}.$$

Lemma 5 (*[3]*). *Let B be a compact convex body in the plane and B_k be a largest area k-gon inscribed in B. Then area$(B_k) \geq$ area$(B)\frac{k}{2\pi}\sin\frac{2\pi}{k}$, where equality holds if and only if B is an ellipse.*

From Theorems 2 and 3, Lemmas 4 and 5, the following results can be easily obtained:

Theorem 6. *For planar point sets in convex position of size 6 and 7 we have*

(1) $\frac{10}{15-\sqrt{5}} \leq f_5^{\text{conv}}(6) \leq r_5(6) = \frac{5}{6}$;
(2) $\frac{5}{10-\sqrt{5}} \leq f_5^{\text{conv}}(7) \leq r_5(7) = \frac{3}{7} + \frac{4}{7}\cos\frac{2\pi}{7}$.

Theorem 7. *For planar point sets in convex position of size $n \geq 8$ we have*

$$\frac{5}{2\pi} \sin \frac{2\pi}{5} \leq f_5^{\text{conv}}(n) \leq r_5(n).$$

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